Method for Analyzing Complex Two-Dimensional Forms: Elliptical Fourier Functions

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ABSTRACT A generalized procedure, elliptical Fourier analysis, for accurately characterizing the shape of complex morphological forms of the type commonly encountered in the biological sciences, is described. Elliptical Fourier functions are derived as a parametric formulation from conventional Fourier analysis, i.e., as a pair of equations that are functions of a third variable. The use of elliptical Fourier functions circumvents three restrictions that have limited conventional Fourier analysis to certain classes of shapes. These restrictions are 1) equal divisions over the interval or period; 2) dependency on the coordinate system, i.e., conventional Fourier functions are not "coordinate free"; and 3) the presence of shapes with outlines that curve back on themselves, a common occurrence. These three limitations are effectively removed with the utilization of elliptical Fourier functions, facilitating the analysis of a much larger class of two-dimensional forms.

Although methods for the accurate description of the shape of complex morphological forms have a long tradition, they remain at present insufficiently developed. This may be due, in part, to the past emphasis on qualitative descriptions in contrast to numerical assessments of form. The emerging use of landmarks in the latter half of the 19th century has aided the organization and simplification of the biological information present in complex forms by facilitating the judicious choice of relevant measurements. However, the conventional metrical approach, which consists of linear distances, angles, and ratios, contains an unavoidable subjective factor, since one is forced to select the measurements to include in an analysis. This is not to suggest that such an endeavor is questionable or useless; it may be entirely appropriate in the context of a particular investigation. However, if the primary concern is an accurate measure of the shape of complex morphological forms, the use of such a procedure is inherently inefficient. For example, the use of angles as "shape measures" has a severe limitation, since the shape lying within an included angle can be, in principle, anything. In other words, the actual shape or outline, presumably of major interest, is not even being measured.

The use of ratios as measures of shape only serves to obscure the presence of actual differences. These problems arise because conventional metrics are designed to measure regular geometric objects, but are being inappropriately applied to describe the shape of complex irregular forms. These issues are by no means new and have been discussed in one form or another by others (Thompson, 1942; Medawar, 1950; Zuckerman, 1950; Corruccini, 1973; Lestrel, 1974, 1982).

The absence of efficient procedures dealing with the shape of irregular forms has focused on anatomical landmarks and thereby has overlooked a fundamental requirement that there be a one-to-one correspondence or mapping between the observed form and the measurements used to describe it. Thus, unless the numerical model is an accurate representation of the form, it is likely that 1) estimates of the variability are probably imprecise, 2) measures of biological distance are distorted, and 3) contributions of the size

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and shape components are being incorrectly ascertained.

What are needed, therefore, are shape descriptors that minimize the subjective element and facilitate the reduction of what may be a large number of observed measurements. The strategy employed here is to embed a set of closely spaced observed measurements on the outline or boundary of a form into a mathematical function. Ideally, what is required of such a function is that it 1) provides an accurate measurement of complex forms, 2) represents a unique solution preferably with orthogonal components, 3) allows for the reduction of the observed data set to a smaller subset of variables, 4) measures a significant percentage of the variability that is present, 5) facilitates the separation of the components of size and shape, 6) can be used to recreate the form with little distortion, 7) is invariant with respect to the coordinate system, 8) applies to two- as well as to three-dimensional data, and 9) relates numerical differences in the coefficients to actual differences in shape.

This paper focuses on two formulations that satisfy the majority of the above criteria. Both methods, the familiar conventional equation as well as the newer elliptic function, are based on Fourier's series. Although the two approaches share aspects, they are computationally different. The conventional Fourier approach (Appendix A) is first described, and the elliptical Fourier function (Appendix B) is developed subsequently. Discussion of the two functions is limited to two-dimensional forms.

**MATERIALS AND METHODS**

The application of the two Fourier functions is demonstrated using craniofacial materials. Lateral cephalometric radiographs were available on a small sample of human adults of a miscellaneous series in the author's possession. They are not intended to be representative. Also available were selected single specimens with which to illustrate the method further. These included a gorilla, Rhodesian Man, and a longitudinal series of a human subject at three ages.

Utilizing a light table, all lateral radiographs were carefully traced onto dimensionally stable acetate sheets using a mechanical pencil with 0.3 mm lead. Once traced, they were digitized using a Houston Instruments Hipad with an accuracy of approximately 0.1-0.2 mm. The digitized data consisting of x and y coordinates were saved to a hard disk file using an IBM XT clone. A specially written routine, in TURBO BASIC, then computed the Fourier functions and generated expected values. The expected values were then used to test the goodness-of-fit of the function to the original observed data points as well as being plotted to provide a visual display. All graphic illustrations in this paper were based on a monochrome monitor with the Hercules graphic card.

**Conventional Fourier analysis**

The development of elliptical Fourier functions can, perhaps, be best understood in the light of conventional Fourier analysis. The applicability of the Fourier series as a biometric fitting function is now well established. Papers in such diverse disciplines as engineering, biology, climatology, geology, archaeology, and physical anthropology are now available (Bliss and Blevins, 1959; Bryson and Dutton, 1961; Kaiser and Halberg, 1962; Lu, 1965; Ehrlich and Weinberg, 1970; Zahn and Roskies, 1972; Kaesler and Waters, 1972; Anstey and Delmet, 1973; Lestrel, 1974; Lestrel et al., 1977, 1986, 1989; Healy-Williams and Williams, 1981; Full and Ehrlich, 1982; Gero and Mazzullo, 1984).

The conventional Fourier series contains both sine and cosine terms. It is often represented by the familiar form

\[
y(t) = A_0 + \sum_{n=1}^{k} a_n \cos(nt) + \sum_{n=1}^{k} b_n \sin(nt).
\]

Here \(k\) is the maximum degree or harmonic number. This is a convergent series that will provide a fit to any known piecewise-smooth function. As the number of terms are added the fit becomes more accurate. Two properties, the interval over which the data points are sampled and the height of the waveform or amplitude, provide a flexible system that will fit many forms. This series is periodic, or "circular," in the sense that it repeats over a set interval. That is, the last data point is followed by the first data point and the process then repeated (see Appendix A).

A useful property of Fourier's series is that the trigonometric relations are pairwise orthogonal on the interval \((\pi, \pi)\). That is, the
integral of the product of any two different trigonometric functions will vanish (Tolstov, 1962). Note that this definition of orthogonality does not connote perpendicularity as is often presumed. Since the coefficients are orthogonal, one can separately assess the contributions that each term in the Fourier series makes toward the approximation of the function, \( y = f(t) \). The formulation of equation 1 is often called the “single” Fourier series, while the extension to three dimensions is termed the “double” Fourier series (for details on the double Fourier series, see Lestrel, 1980).

If Fourier’s series (eq. 1) is converted to polar coordinates (Appendix A), then the formulation has the additional property of being able to relate the numerical differences in the first few harmonics with actual differences in the shape under consideration (Lestrel, 1974). The polar coordinate form is shown as

\[
r = f(\theta) = A_0 + \sum_{i=1}^{k} a_i \cos(i\theta) + \sum_{i=1}^{k} b_i \sin(i\theta). \tag{2}
\]

Besides the requirement of a function that accurately fits the sampled data points on the boundary of an outline, there are two other critical considerations that need to be resolved before complex biological shapes can be meaningfully compared. These are the normalization procedures, termed **positional-orientation** and **size-standardization**. For example, positional-orientation in polar coordinates requires that the center from which the vectors are measured is common to all forms, because the harmonic amplitudes are dependent on this center. Change the origin, and the amplitude values change. This dilemma can be resolved with the use of a neutral center such as the centroid (which is invariant with rotation). The translation to the centroid is computed via a recursive process to ensure that equal angular intervals are preserved and new vectors computed to the observed points on the boundary. The overriding consideration is that the shape as measured by the original vectors is faithfully maintained. In polar coordinates, the translation to the centroid has the effect of making the first harmonic in the Fourier function vanish. If there are no landmarks, no recognizable features for orientation, then rotation about the centroid to minimize differences in shape may be the only viable solution, in effect a cross-correlation procedure.

The second normalization procedure is the correction for differences in size. This is predicated on the grounds that the presence of substantial differences in size may serve to overwhelm differences in shape. There are two ways of dealing with bounded outlines: 1) A scaling factor based on the constant, \( A_0 \), and 2) a scaling factor based on the actual area under the bounded form. The constant, \( A_0 \), can be defined (in polar coordinates) as the mean of the vectors from the centroid to the outline of the form, so that it can readily be used as the scaling factor and the remainder of the coefficients adjusted by multiplication of the factor, constant/\( A_0 \). Although rapidly computed, it has the drawback that if the forms being compared differ considerably in shape, this scaling factor becomes inaccurate. The second scaling factor, based on the area, while computationally more involved, is preferable (for details on these normalizations, see Parnell and Lestrel, 1977; Lestrel, 1980).

In spite of the successful application of equations 1 and 2 to a wide variety of skeletal data (Ehrlich and Weinberg, 1970; Anstey and Delmet, 1973; Lestrel and Brown, 1976; Lestrel and Roche, 1976, 1979; Lestrel et al., 1977; Lestrel and Moore, 1978; Lestrel and Sirianni, 1982; Gero and Mazzullo, 1984; Lestrel et al., 1986, 1989), a number of restrictions have limited the types of forms that can be successfully characterized. These constraints are 1) equal divisions (angular ones if in polar coordinates) over the interval; 2) the use of a center and a starting point on the boundary that results in a dependency on the coordinate system, i.e., the method is not “coordinate free”; and 3) the presence of complex shapes requiring multivalued functions for a satisfactory fit. The latter arise when outlines curve back on themselves. An example of this is illustrated in Figure 1, which was taken from a previous study of the cranial base (Lestrel and Roche, 1986). The lateral endocranial outline is depicted here from basion to the anterior cranial base in the vicinity of *Crista galli*. These two specimens differ in size. Note that the expected fit (dotted lines) using equation (A-13 in Appendix A) is poor at the posterior aspect of the dorsal clivus.
This arises because in this region of the cranial base a double-valued function would be required. That is, a vector drawn from the origin, VCTR, would have to have two lengths, one to the inner margin and one to the dorsal clivus. This is inadmissible without casting part of the measurement set into the imaginary plane, resulting in an undesirable increase in computational complexity and subsequent interpretation. These limitations have restricted the use of conventional Fourier analysis to certain classes of two-dimensional shapes.

Recent objections to the use of Fourier analysis (Bookstein et al., 1982) have been answered elsewhere (Ehrlich et al., 1983; Read and Lestrel, 1986). The ensuing controversy has arisen because of the differences between the use of biorthogonal grids, which, broadly interpreted, is a landmark dependent system while Fourier analysis may be viewed as a “landmark-free” system. There are advantages and limitations to both approaches, since all methods have constraints. It is apparent that no single method encompasses both the homologous point information (landmarks) and the boundary curve information (outline) into a single numerical model. A first tentative step in this direction is offered by Read and Lestrel (1986).

The transition to elliptical Fourier functions (Kuhl and Giardina, 1982) circumvents some of the limitations of the conventional Fourier series, thereby facilitating an analysis of a much larger class of two-dimensional shapes (Rohlf and Archie, 1984; Ferson et al., 1985).

**Elliptical Fourier analysis**

The development of elliptic Fourier functions by Kuhl and Giardina (1982) introduces a new approach to the old problem of numerically characterizing complex shapes. Basically, elliptic Fourier functions represent a parametric solution, i.e., the derivation of a pair of equations as functions of a third variable. Any two-dimensional outline can be approximated with a polygon by connecting the observed data points with straight lines. Figure 2 is an example of a maxilla, including the upper central incisor, as viewed on a lateral radiograph, that has been approximated with such a polygon. The closer the spacing of the observed points, the more accurate the representation. Note that the distances between data points can vary. If these points can be thought of as traveling along the outline with constant speed, then the projection of these points in either the x- or the y-axis can also be defined as a function of time. Since these new functions are piecewise linear, single-valued, and periodic, they can also be fitted with Fourier functions (Kuhl and Giardina, 1982). Consider the y-coordinate values of Figure 2 being projected or “embedded” into the y-axis and then replotted against a new horizontal “x-axis,” which has been labeled the “t-axis” (the time axis). The x-coordinate values are treated in an identical manner. These x and y “projections” are shown in Figures 3 and 4. Note that although the order of the x- and y-coordinates cannot be altered, the size of the equal divisions over the interval on the t-axis can be arbitrary. Examination of these figures reveals that they will always be single-valued and periodic. It is emphasized that it is Figures 3 and 4 that are being fitted with the Fourier functions, not the original two-dimensional form. Once the expected x- and y-coordinates have been separately computed, one can “rejoin” these coordinates (for identical values of t) and the expected shape recreated. If the separate harmonics are plotted (note that they are orthogonal), they produce ellipses. As the ellipses are summed in an identical fashion to the conventional Fourier function already
discussed, they will converge onto the polygon that serves as the observed form.

The parametric equations are defined in \( x(t) \) as

\[
x_p = f(t) = A_o + \sum_{n=1}^{k} a_n \cos(nt) + \sum_{n=1}^{k} b_n \sin(nt)
\]

and in \( y(t) \) as

\[
y_p = f(t) = C_o + \sum_{n=1}^{k} c_n \cos(nt) + \sum_{n=1}^{k} d_n \sin(nt).
\]

Inspection of equations 3 and 4 shows that they are identical to the nonparametric case, equation 1, outlined earlier. The derivation of these equations is developed in Appendix B.

As with conventional Fourier analysis, forms have to be normalized for size and orientation before meaningful comparisons can be realized. Figure 5 displays four theoretically possible normalizations involving 1) orientation such as translation and/or rotation and 2) size adjustment or scaling. By no positional orientation (Fig. 5A,C), this implies that we have defined the orientation, e.g., on the Sella-Nasion plane or perhaps on some other line based on biological criteria. Such an orientation cannot be arbitrary from specimen to specimen, because the elliptical Fourier coefficients are dependent on the coordinate system much the same way as the conventional Fourier coefficients discussed previously. However, Kuhl and Giardina (1982) have offered a way to use the form itself to generate an internal orientation for comparison. They suggest rotating the form until the major axis of the first harmonic ellipse is parallel to an axis, as shown in Figure 5B,D.

Besides the problem of orientation, there also remains the need to normalize for differences in size. Kuhl and Giardina (1982) have recommended scaling the length of the semimajor axis of the first harmonic ellipse so that it is equal to 1 and adjusting the remaining coefficients accordingly. This is shown schematically in Figure 5C,D, where the two illustrations are drawn slightly smaller to reflect this correction in size. This is analogous to using the constant, \( A_o \), of the conventional Fourier function as a scaling factor and suffers from the same difficulties.
Fig. 3. Cartesian coordinate plot (x,t) of the maxilla shown in Figure 2 in which the x-coordinate values (x-projections) are plotted as a function of a third variable, t, on the horizontal axis (see text).

Fig. 4. Cartesian coordinate plot (y,t) of the maxilla shown in Figure 2 in which the y-coordinate values (y-projections) are plotted as a function of a third variable, t, on the horizontal axis (see text).

discussed earlier. A more appropriate solution to the scaling problem is to utilize the area bounded by the polygon.

Thus, three limitations present with the use of conventional Fourier functions have been removed: 1) The requirement of equal divisions over the interval or period; 2) the presence of shapes requiring multivalued functions to ensure a satisfactory fit, since elliptical Fourier functions will always be single-valued; and 3) the ability to use the form itself to generate an internal orientation system for comparison with other forms.

RESULTS

To illustrate visually the power of elliptical Fourier functions, a number of radiographic specimens were utilized. Although these are, for the most part, individual specimens,
samples are being prepared for subsequent publication. The illustrations presented here are intended to demonstrate the methodology.

Figure 6 is a computer-generated graphic display of the human cranial base, with the 54 points describing the outline as traced from a lateral head film. The locations of the 54 data points are enclosed within circles to provide easier identification. The elliptical Fourier fit was truncated at 20 harmonics, which was considered satisfactory, since the residual value was less than 0.20 mm, well below the joint errors of tracing and digitizing. The poorest fit occurred at Basion, where the sharpest change in curvature was present, but even here the residual did not rise above 0.65 mm. Note that with 20 harmonics, there are 80 separate terms, four for each harmonic (the $a_n$, $b_n$, $c_n$, and $d_n$ coefficients) as well as the two constants ($A_0$ and $C_0$).

Figure 7 is the elliptical Fourier fit to the same cranial base showing the convergence of the elliptical Fourier function as a stepwise procedure. The first harmonic represents an ellipse as shown in the upper left drawing in Figure 7. With the addition of 10 harmonics, the residual fit, or difference between the observed data points and the predicted value from the elliptical Fourier function, was 0.6 mm overall. With 30 harmonics the mean residual dropped to less than 0.13 mm.

Generated as part of an ongoing study of the shape of the cranial base, Figure 8 is an amplitude versus harmonic number plot of a cranial base sample of human adults com-
Fig. 6. Computer-generated graphic display of the outline of the human cranial base traced from a lateral cephalometric radiograph and described with 54 points. The actual points are enclosed within circles for clarity. The outline has been fitted with the elliptical Fourier function using 20 harmonics. The data are not normalized for size or orientation.

Fig. 7. Lateral view of the same cranial base as shown in Figure 6. The elliptical Fourier fit is shown as a stepwise process to illustrate the convergence of the series to the actual form under consideration.
pared with a single gorilla specimen. The amplitude plot of the x-coordinate data is shown in Figure 8A, while the amplitude plot of the y-axis data is depicted in Figure 8B. Every specimen was normalized for orientation by superimposing each cranial base on the major axis of the first harmonic ellipse and for size by making the semimajor axis equal to 1 according to the procedures advocated by Kuhl and Giardina (1982). Although, the size-standardization may only be approximate, the harmonic amplitude plot shows that distinct differences are obtained in the comparison.

Figure 9 represents an attempt to describe numerically the complete craniofacial complex consisting of maxilla, mandible, cranial base, and cranial vault. A system of 330 points was devised to characterize the head as a two-dimensional polygon. The observed data were traced from a lateral radiograph. In areas of abrupt change in curvature, the points are closer together. Utilizing an elliptical Fourier function with 100 harmonics yielded an average residual of less than 0.50 mm for the whole cranium. The head is thus numerically described using a pair of equations that together consist of 402 separate terms. The original observed data points could be discarded, since the cranium outline can be recreated at any time using the elliptical Fourier function.

Given the large number of data points and the extensive number of harmonics that need to be computed to affect an acceptable fit, the question necessarily arises as to cost...
in terms of time. The fit of this function, with 330 data points and 100 harmonics, took less than 2 minutes of CPU time on an IBM XT clone running at 8 mHz using TURBO BASIC with a 8087 coprocessor. These times could be reduced to seconds with the faster machines currently available. This indicates that the Kuhl and Giardina algorithm is quite efficient for the computation of large data sets of this type.

In an attempt to simplify the orientation, a procedure used with considerable success with the conventional Fourier function, was also applied here. Figure 10 depicts the

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Fig. 9. Computer-generated display of the complete craniofacial complex composed of maxilla, mandible, cranial vault, and cranial base as traced from a lateral head film. The head was described with 330 points and fitted with an elliptical Fourier function using 100 harmonics.

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Fig. 10. The cranial vault flipped about the Sella-Nasion plane to create a “mirror image.” While this procedure imposes symmetry and requires a doubling of the observed data points, it leads to a simplified elliptical Fourier function as the sine terms vanish and provides a common orientation for all specimen comparisons.
mirror-imaging approach, applied to the cranial vault, which results in a symmetric form. It will be recalled that if symmetry can be imposed, then only even terms will be present and the odd terms will vanish (Les-
trel, 1974).

Figure 11 depicts an amplitude versus harmonic number plot of the comparison of three single specimens: An adult male gorilla, Rhodesian Man (a Neandertholoid specimen perhaps 125,000 years old), and a modern Homo sapiens adult male. The three specimens have been superimposed on the Sella-Nasion plane. Because of the imposed mirror-image, the major axis of the first harmonic ellipse is also parallel to this plane. These specimens have been size-standardized such that the semimajor axis of the first harmonic ellipse is equal to 1. As these are single specimens, there is no ques-
tion of making any phylogenetic inferences. This comparison is only intended to illustrate the method. Figure 11A is a plot of the x-axis, and Figure 11B represents the data for the y-axis. Major differences in the cranial vault outlines are evident for both axes. The first few harmonics account for most of the variability in overall shape, or “bau-
plan,” while the higher harmonics reflect the fine detail, or localized sculpturing, of the vault outline. Figure 11C has the same data as in Figure 11A but the vertical axis is magnified about fivefold. It shows the presence of considerable variability in all the harmonics. Figure 11D reflects the same pattern of considerable variability in cranial vault shape for the y-axis.

Finally, Figure 12A,B depicts amplitude versus harmonic number plots of a longitudinal comparison of a single subject, namely, as a 2 year-old child, as a 8 year-old juvenile, and as a 18-year-old adult. All three tracings were superimposed on the Sella-Nasion plane and mirror-imaged. These data were then normalized for size in an identical way to the data displayed in Figure 11. Note that the amplitude versus harmonic number plot for both the x- and the y-axes (Fig. 12A,B) is

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Fig. 11. Amplitude versus harmonic number plot for the cranial vaults of three specimens: an adult male gorilla, Rhodesian Man, and a modern adult male. These specimens have been mirror-imaged and normalized for size. A: Amplitude plot for the x-coordinate data. B: Amplitude plot for the y-coordinate data. C: Amplitude plot for the x-coordinate data magnified fivefold. D: Amplitude plot for the y-coordinate data magnified fivefold (see text).
quite similar for all three ages, which is not unexpected. Again, if the vertical axis is magnified fivefold (Fig. 12C,D) differences begin to be apparent, although the magnitude of these differences is quite small. There is a hint of more variability in the y-axis, but this needs to be verified with a sample. The examples presented here are only intended to illustrate the applicability of elliptical Fourier functions as a method for characterizing the shape of complex two-dimensional forms.

CONCLUSIONS

The basic theme of this paper has dealt with the need for numerical methods that can accurately characterize the shape of complex morphological shapes and how, once characterized, such forms can be meaningfully compared. As stated previously, this issues revolves around the relationship between the measurements used and the morphological form to be described. In other words, what is required is a one-to-one mapping of the measurement domain with the form domain. Only when a significant proportion of the information present in all forms is measured will it be possible to identify those aspects that presumably make a major contribution to the shape. Conventional Fourier analysis has been successfully applied to a variety of data sets. While it is comparatively easy to compute and apply, it is limited to certain classes of simple two-dimensional forms. The utilization of elliptical Fourier analysis, on the other hand, removes three of the limitations present with conventional Fourier analysis. These are equal divisions over the interval sampled, dependency on the coordinate system, and the difficulty of dealing with outlines that curve back on themselves. Elliptical Fourier functions represent a powerful new approach for extracting a significant percentage of biological information that is
readily visualized and yet difficult to quantify with conventional methods.

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LITERATURE CITED

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APPENDIX A: CONVENTIONAL FOURIER ANALYSIS

The conventional Fourier series contains both sine and cosine terms. The generalized infinite Fourier series is of the form

\[ y = f(t) = A_o + \sum_{n=1}^{\infty} a_n \cos(2\pi nt/T) + \sum_{n=1}^{\infty} b_n \sin(2\pi nt/T), \quad (A-1) \]

where \( y \), the dependent variable, is the height of the wave, \( A_o \) is the constant, \( n \) is the degree or harmonic number, \( t \) is the sampling points along the \( t \) axis, and \( T \) is of the fundamental sampling length or period. Note that the period need not correspond with the usual interval of 0 to 2\( \pi \). This Fourier function can be graphed as a \( y \) versus \( t \) plot in cartesian coordinates where the \( t \)-axis is a time axis. The use of a “time axis” will be made clearer in the next section. If the data are spatial, not time dependent, then the \( t \)-axis would be replaced by the \( x \)-axis, the usual case. The derivations are identical whether the data are dependent on time or are spatially oriented.

If the function \( y = f(t) \) is known, i.e., for which a mathematical function exists, then the constant, \( A_o \), and the coefficients, \( a_n \) and \( b_n \), are evaluated from the integrals:

\[ A_o = 1/T \int_0^t f(t) \, dt, \quad (A-2) \]

\[ a_n = 2/T \int_0^t f(t) \cos(2\pi nt/T) \, dt, \quad (A-3) \]

and

\[ b_n = 2/T \int_0^t f(t) \sin(2\pi nt/T) \, dt, \quad (A-4) \]

where the period or interval is from 0 to \( T \), and \( n \) is the harmonic number.

If the period is defined over a 2\( \pi \) interval, i.e., where \( T = 2\pi \), and limits changed, equations A-1 simplifies to the familiar finite form

\[ y = f(t) = A_o + \sum_{n=1}^{k} a_n \cos(nt) + \sum_{n=1}^{k} b_n \sin(nt), \quad (A-5) \]

shown in the text as equation 1. If the period is defined over the \((-\pi, \pi)\) interval, then the constant, \( A_o \), and the \( a_n \) and \( b_n \) coefficients of equation A-5 are evaluated from the integrals

\[ A_o = 1/2\pi \int_{-\pi}^{\pi} f(t) \, dt, \quad (A-6) \]

\[ a_n = 1/\pi \int_{-\pi}^{\pi} f(t) \cdot \cos(nt) \, dt, \quad (A-7) \]

and

\[ b_n = 1/\pi \int_{-\pi}^{\pi} f(t) \cdot \sin(nt) \, dt. \quad (A-8) \]

If the function, \( y = f(t) \), is unknown, the usual case with biological data of the type under consideration here, this precludes an analytical solution using equations A-2, A-3, and A-4, or A-6, A-7, and A-8, and recourse must be made to numerical integration procedures such as the trapezoidal rule. In this case the observed data would be tabulated as \( x_n, y_n \) or, as in the above case, \( t_n, y_n \). It is of interest that the trapezoidal rule satisfies the least-squares criterion of minimizing the sum of the squared deviations. If \( p \) is the total number of points chosen with equal divisions over the interval \((-\pi, \pi)\), which precludes the need for a weighted analysis, then the general expressions for the evaluation of the constant, \( A_o \), and the \( a_n \) and \( b_n \) coefficients are derived from

\[ A_o = \sum_{i=1}^{p} f(t_i)/p, \quad (A-9) \]

\[ a_n = 2/p \sum_{i=1}^{p} f(t_i) \cdot \cos(nt_i), \quad (A10) \]

and

\[ b_n = 2/p \sum_{i=1}^{p} f(t_i) \cdot \sin(nt_i), \quad (A-11) \]
where \( p \) is the total number of points, \( Y_i = f(t_i) \), and \( n \) is the harmonic number as before. Note that the constant, \( A_0 \), is the mean of the observed \( y_i \) values. There is no B_s term. These solutions are subject to the Nyquist sampling rate, which requires that the maximum number of computed harmonics does not exceed 50% of the number of observed data points (for a detailed discussion of these equations, see Harbaugh and Preston, 1968; Harbaugh and Merriam, 1968; Davis, 1973).

Again, restricting the subdivisions within the period to equal intervals and changing the notation to reflect the conversion to polar coordinates \( r, \theta \) yields

\[
r = f(\theta) = A_0 + \sum_{i=1}^{k} a_i\cos(i\theta) + \sum_{i=1}^{k} b_i\sin(i\theta), \quad (A-12)
\]

where \( \theta \) is in radians, \( r \) is the predicted range to the outline from a predetermined center, \( k \) is the maximum harmonic number, and the period is again defined over a \( 2\pi \) interval. Evaluation of the constant, \( A_0 \), and the \( a_i \) and \( b_i \) coefficients is via the numerical integration methods outlined in equations A-9, A-10, and A-11, but requiring the notational change to reflect polar coordinates.

The \( a_i\cos(i\theta) \), or even terms, describe symmetric patterns, while the \( b_i\sin(i\theta) \), or odd terms, express asymmetry. If the form under consideration is symmetric or if symmetry can be imposed, then the sine terms will vanish and equation A-12 will reduce to

\[
r = f(\theta) = A_0 + \sum_{i=1}^{k} a_i\cos(i\theta). \quad (A-13)
\]

Expansion of equation A-13 yields the series

\[
r = A_0 + a_1\cos(\theta) + a_2\cos(2\theta) + \cdots + a_k\cos(k\theta). \quad (A-14)
\]

This formulation has the additional property of being able to relate numerical differences in the first few harmonics with actual differences in the shape under consideration (Lestrel, 1974). Finally, it must be cautioned that the extension to three dimensions is not straightforward in polar coordinates in contrast to the cartesian system, because the property of orthogonality breaks down (Lestrel, 1980).

APPENDIX B:

ELLIPTICAL FOURIER FUNCTIONS

The parametric equations are defined such that the Fourier series in \( x(t) \) is given as

\[
x_p = f(t) = A_0 + \sum_{n=1}^{k} a_n\cos(nt) + \sum_{n=1}^{k} b_n\sin(nt) \quad (B-1)
\]

and the Fourier series in \( y(t) \) as

\[
y_p = f(t) = C_0 + \sum_{n=1}^{k} c_n\cos(nt) + \sum_{n=1}^{k} d_n\sin(nt), \quad (B-2)
\]

where \( n \) equals the harmonic number, \( k \) equals the maximum harmonic number, and the interval is over \( 2\pi \), as before. Inspection of equations B-1 and B-2 shows that they are identical to the nonparametric case, equation 1 described in the text and equation A-5 in Appendix A.

As stated in the text, if the sampled points along the polygon can be thought of traveling at constant speed, then the first derivative, \( \dot{x}_p = \frac{dx_p}{dt} \) or \( \dot{y}_p = \frac{dy_p}{dt} \), is also piecewise-constant and can be represented by a Fourier series. These derivatives also consist of a sequence of piecewise-constant derivatives, \( \Delta x_p / \Delta t_p \) (or \( \Delta y_p / \Delta t_p \)), associated with each division (since they are straight lines), along the polygon, \( t_p \) to \( t_p+1 \), over the period \( T \), where \( \Delta t_p = (\Delta x_p^2 + \Delta y_p^2)^{1/2} \). Utilizing these properties, Kuhl and Giardina (1982) derived estimates for the elliptical coefficients that do not require integration. Simplifying by setting the period \( T = 2\pi \) yields the Fourier coefficients for the \( x \)-projection as

\[
a_n = 1/n^2\pi \sum_{p=1}^{q} \frac{\Delta x_p / \Delta t_p [\cos(nt_p) - \cos(nt_{p-1})]}{\sin(nt_{p-1})}, \quad (B-3)
\]

and

\[
b_n = 1/n^2\pi \sum_{p=1}^{q} \frac{\Delta x_p / \Delta t_p [\sin(nt_p) - \sin(nt_{p-1})]}{\sin(nt_{p-1})}, \quad (B-4)
\]
where \( q \) is the total number of \( p \) points along the polygon, \( n \) is the harmonic number, \( t_p \) is the distance between point \( p \) and point \( p+1 \) along the polygon, and \( \Delta x_p \) and \( \Delta y_p \) are the respective projections of \( p \) to \( p+1 \).

The Fourier coefficients for \( y \)-projections are

\[
c_n = 1/n^2 \pi \sum_{p=1}^{q} \Delta y_p/\Delta t_p \left[ \cos(nt_p) - \cos(nt_{p-1}) \right] \tag{B-5}
\]

and

\[
d_n = 1/n^2 \pi \sum_{p=1}^{q} \Delta y_p/\Delta t_p \left[ \sin(nt_p) - \sin(nt_{p-1}) \right] \tag{B-6}
\]

Besides the four coefficients, \( a_n, b_n, c_n, \) and \( d_n \), two constants, \( A_0 \) and \( C_0 \), need to be estimated. In an analogous way to the conventional Fourier function, \( B_0 = D_0 = 0 \). The evaluation of the \( A_n \) and \( C_n \) terms, which can be considered weighted \( x \) and \( y \) coordinates of the center of the form, are computed from

\[
A_0 = \frac{1}{2\pi} \sum_{p=1}^{q} \frac{\Delta x_p/\Delta t_p}{2t_p} \left[ t^2 - t_{p-1}^2 \right] + \frac{\alpha_p}{t_p - t_{p-1}}, \tag{B-7}
\]

and

\[
C_0 = \frac{1}{2\pi} \sum_{p=1}^{q} \frac{\Delta y_p/\Delta t_p}{2t_p} \left[ t^2 - t_{p-1}^2 \right] + \frac{\beta_p}{t_p - t_{p-1}}, \tag{B-8}
\]

The \( \alpha_p \) and \( \beta_p \) terms are given as

\[
\alpha_p = \sum_{j=1}^{p-1} \Delta x_j - \left[ \Delta x_p/\Delta t_p \sum_{j=1}^{p-1} \Delta t_j \right] \tag{B-9}
\]

and

\[
\beta_p = \sum_{j=1}^{p-1} \Delta x_j - \left[ \Delta x_p/\Delta t_p \sum_{j=1}^{p-1} \Delta t_j \right] \tag{B-10}
\]

where

\[
\alpha_1 = \beta_1 = 0.
\]